

# Integration with respect to local time and Itô's formula for smooth nondegenerate martingales

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## Abstract

We show an Itô's formula for nondegenerate Brownian martingales  $X_t = \int_0^t u_s dW_s$  and functions  $F(x, t)$  with locally integrable derivatives in  $t$  and  $x$ . We prove that one can express the additional term in Itô's formula as an integral over space and time with respect to local time.

**Keywords:** Martingales; Integration wrt local time; Itô's formula; Local time;

**Running title:** Itô's formula for nondegenerate martingales

## Introduction

We consider a continuous nondegenerate martingale  $X = \{X_t, t \in [0, 1]\}$  of the form  $X_t = \int_0^t u_s dW_s$  where  $W = \{W_t, t \in [0, 1]\}$  is a standard Brownian motion and  $u$  is an adapted stochastic process. Let  $F : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$  be an absolutely continuous function with partial derivatives satisfying some local integrability properties. The main aim of this paper is to obtain an Itô's formula for  $F(X_t, t)$  where the term corresponding usually to the second order derivative is expressed as an integral over space and time with respect to local time.

We will prove this results when  $u$  satisfies (locally) the assumptions

**(H1)** For all  $t \in [0, 1]$ ,  $u_t$  belongs to the space  $\mathbb{D}^{3,2}$  and for all  $p \geq 2$

$$E|u_t|^p + E|D_s u_t|^p + E\left(\int_{r \vee s}^1 |D_r D_s u_\theta|^2 d\theta\right)^{p/2} + E\left(\int_{r \vee s \vee v}^1 |D_v D_r D_s u_\theta|^2 d\theta\right)^{p/2} \leq K_p,$$

**(H2)**  $|u_t| \geq \rho > 0$  for some constant  $\rho$  and for all  $t \in [0, 1]$ .

Moret and Nualart (2000) consider an Itô's formula for this class of nondegenerate martingales. Their main result reads as follows:

**Theorem 0.1 (Moret and Nualart, 2000)** *Let  $u$  be a process satisfying (H1) and (H2). Set  $X = \int_0^t u_s dW_s$ . Then for any function  $f \in L^2_{\text{loc}}(\mathbb{R})$  the quadratic covariation  $[f(X), X]$  exists and the following Itô's formula holds*

$$F(X_t) = F(0) + \int_0^t f(X_s) dX_s + \frac{1}{2}[f(X), X]_t,$$

for all  $t \in [0, 1]$ , where  $F(x) = F(0) + \int_0^x f(y) dy$ .

Moret (1999), gave an extension of this last result for functions  $F$  depending also on  $t$ . They consider a new hypothesis on functions  $f$ :

(C)  $f(\cdot, t) \in L^2_{loc}(\mathbb{R})$  and for all compact set  $K \subset \mathbb{R}$   $f(x, t)$  is continuous in  $t$  as a function of  $[0, T]$  to  $L^2(K)$

Then, their result is the following:

**Theorem 0.2 (Moret, 1999)** *Let  $u$  be a process satisfying (H1) and (H2). Set  $X = \int_0^t u s dW_s$ . Let  $F(x, t)$  be an absolutely continuous function in  $x$  such that the partial derivative  $f(\cdot, t)$  satisfies (C). Then, the quadratic covariation  $[f(X, \cdot), X]$  exists and the following Itô's formula holds*

$$F(X_t, t) = F(0, 0) + \int_0^t f(X_s, s) dX_s + \frac{1}{2} [f(X, \cdot), X] + \int_0^t F(X_s, ds),$$

where

$$\int_0^t F(X_s, ds) \equiv \lim_{n \rightarrow +\infty} \sum_{t_i \in D_n, t_i \leq t} (F(X_{t_{i+1}}, t_{i+1}) - F(X_{t_{i+1}}, t_i)),$$

exists uniformly in probability for  $(D_n)_n$  a sequence of smooth partitions of  $[0, 1]$ .

In these two results, following the ideas of Föllmer, Protter and Shiryaev (1995) for the Brownian motion, the additional term is written as a quadratic covariation. Bardina and Jolis (1997, 2002) extended the results of Föllmer *et al.* (1995) to the case of the elliptic and hypoelliptic diffusions.

Nevertheless, it is important to point out the differences between the work of Moret and Nualart (2000) and Föllmer *et al.* (1995). One of the keys of their proofs is to obtain some a priori estimates on the Riemann sums. In Föllmer *et al.* (1995) these estimates are obtained using the semimartingale expression of the time-reversed Brownian motion and well-known bounds for the density of the Brownian motion. Moret and Nualart (2000) used another approach, using Malliavin calculus in order to obtain sharp estimates for the density of the process  $X_t$  and avoiding the time-reversed arguments.

We want to express the quadratic variation term as an integral with respect to the local time. There are several papers where the integrals with respect to local time are used in Itô's formula. In 1981, Bouleau and Yor obtained the following extension of the Itô's formula :

**Theorem 0.3 (Bouleau and Yor, 1981)** *Let  $X = (X_t)_{t \geq 0}$  be a continuous semimartingale and let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be an absolutely continuous function with derivative  $f$ . Assume that  $f$  is a measurable locally bounded function. Then:*

$$F(X_t) = F(X_0) + \int_0^t f(X_s) dX_s - \frac{1}{2} \int_{\mathbb{R}} f(x) d_x L_t^x$$

where  $d_x L_t^x$  is an integral with respect to  $x \rightarrow L_t^x$ .

Eisenbaum (2000, 2001) defined an integral in time and space with respect to the local time of the Brownian motion. Using this integral, the quadratic covariation in the formula given in Föllmer *et al.* can be expressed as an integral with respect to the local time. She obtained the following result:

**Theorem 0.4 (Eisenbaum, 2000 and 2001)** *Let  $W = (W_t)_{0 \leq t \leq 1}$  be a standard Brownian motion and  $F$  a function defined on  $\mathbb{R} \times [0, 1]$  such that there exist first order Radon-Nikodym derivatives  $\frac{\partial F}{\partial t}$  and  $\frac{\partial F}{\partial x}$  such that for every  $A \in \mathbb{R}_+$ ,*

$$\int_0^1 \int_{-A}^A \left| \frac{\partial F}{\partial t}(x, s) \right| \frac{1}{\sqrt{s}} dx ds < +\infty$$

and

$$\int_0^1 \int_{-A}^A \left( \frac{\partial F}{\partial x}(x, s) \right)^2 \frac{1}{\sqrt{s}} dx ds < +\infty.$$

Then,

$$F(W_t, t) = F(W_0, 0) + \int_0^t \frac{\partial F}{\partial x}(W_s, s) dW_s + \int_0^t \frac{\partial F}{\partial t}(W_s, s) ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) dL_s^x.$$

This result has been extended by Bardina and Rovira (2007) for elliptic diffusion processes.

In our papers we will follow the ideas Eisenbaum (2000, 2001), assuming on the function  $F$  the hypothesis considered in Theorem 0.4. In the papers of Eisenbaum (2000, 2001), as well as in Föllmer *et al.* (1995) or in the extension of Bardina and Rovira (2007), one of the main ingredients is the study of the time reversed process and the relationship between the quadratic covariation and the forward and backward stochastic integrals. We show that we can adapt the methods of Eisenbaum without using the time reversed process and the backward integral. We will follow the methods of Moret and Nualart (2000) and we will use Malliavin calculus to obtain the necessary estimates for the Riemann sums.

In our paper, the existence of the quadratic covariation is not one of our main objectives. Nevertheless, it will be an important tool in our computations. We recall its definition.

**Definition 1** *Given two stochastic processes  $Y = \{Y_t, t \in [0, 1]\}$  and  $Z = \{Z_t, t \in [0, 1]\}$  we define their quadratic covariation as the stochastic process  $[Y, Z]$  given by the following limit in probability, if it exists,*

$$[Y, Z]_t = \lim_n \sum_{t_i \in D_n, t_i < t} (Y_{t_{i+1}} - Y_{t_i})(Z_{t_{i+1}} - Z_{t_i}).$$

where  $D_n$  is a sequence of partitions of  $[0, 1]$ .

We will assume that the partitions  $D_n$  satisfy

$$(M) \quad \lim_n \sup_{t_i \in D_n} (t_{i+1} - t_i) = 0, \quad M := \sup_n \sup_{t_i \in D_n} \frac{t_{i+1}}{t_i} < \infty.$$

We impose this condition in order to avoid certain possibly exploding Riemann sums.

Other extensions for Itô's formula has been obtained recently. Among others, there is the paper of Dupoirion *et al.* (2004) for uniformly elliptic diffusions and Dirichet processes, the work of Ghomrasni and Peskir (2006) for continuous semimartingales, the paper of Flandoli, Russo and Wolf (2004) for a Lyons-Zheng process or the work of Di Nunno, Meyer-Brandis, Øksendal and Proske (2005) for Lévy processes.

The paper is organized as follows. In Section 1 we give some basic definitions and results on Malliavin calculus, recalling some results obtained in Moret and Nualart (2000). In Section 2 we

define the space where we are able to construct an integral in the plane with respect to the local time of a nondegenerate Brownian martingale. Finally, Section 3 is devoted to present our main result the extension of Itô's formula.

Along the paper we will denote all the constants by  $C, C_p, K$  or  $K_p$ , unless they may change from line to line.

## 1 Preliminaries

Let  $(\Omega, \mathcal{F}, P)$  be the canonical probability space of a standard Brownian motion  $W = \{W_t, 0 \leq t \leq 1\}$ , that is,  $\Omega$  is the space of all continuous functions  $\omega : [0, 1] \rightarrow \mathbb{R}$  vanishing at 0,  $P$  is the standard Wiener measure on  $\Omega$  and  $\mathcal{F}$  is the completion of the Borel  $\sigma$ -field of  $\Omega$  with respect to  $P$ . Let  $H = L^2([0, 1])$ .

Let  $\mathcal{S}$  be the set of smooth random variables of the form

$$F = f(W_{t_1}, \dots, W_{t_n}), \quad (1)$$

$f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$  and  $t_1, \dots, t_n \in [0, 1]$ . The Malliavin derivative of a smooth random variable  $F$  of the form (1) is the stochastic process  $\{D_t F, t \in [0, T]\}$  given by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W_{t_1}, \dots, W_{t_n}) I_{[0, t_i]}(t), \quad t \in [0, 1].$$

The Malliavin derivative of order  $N \geq 2$  is defined by iteration, as follows. For  $F \in \mathcal{S}$ ,  $t_1, \dots, t_N \in [0, T]$ ,

$$D_{t_1, \dots, t_N}^N F = D_{t_1} D_{t_2} \dots D_{t_N} F.$$

For any real number  $p \geq 1$  and any integer  $N \geq 1$  we denote by  $\mathbb{D}^{N,p}$  the completion of the set  $\mathcal{S}$  with respect to the norm

$$\|F\|_{N,p} = \left[ E(|F|^p) + \sum_{i=1}^N E(\|D^i F\|_{L^2([0, T]^i)}^p) \right]^{\frac{1}{p}}.$$

The domain of the derivative operator  $D$  is the space  $\mathbb{D}^{1,2}$ .

The divergence operator  $\delta$  is the adjoint of the derivative operator. The domain of the operator  $\delta$ , denoted by  $\text{Dom } \delta$ , is the set of processes  $u \in L^2([0, T] \times \Omega)$  such that there exists a square integrable random variable  $\delta(u)$  verifying

$$E(F\delta(u)) = E\left(\int_0^1 D_t F u_t dt\right),$$

for any  $F \in \mathcal{S}$ . The operator  $\delta$  is an extension of Itô's stochastic integral and we will make use of the notation  $\delta(u) = \int_0^1 u_s dW_s$ .

We will recall some useful results from Moret and Nualart (2000). We refer the reader to this paper for their proof and a detailed account of these results. We also refer to Nualart (1995, 2006) for any other property about operators  $D$  and  $\delta$ .

**Proposition 1.1** *Let  $Y$  be a random variable in the space  $\mathbb{D}^{1,2}$  such that  $\int_a^b (D_s Y)^2 ds > 0$  a.s. for some  $0 \leq a < b \leq 1$ . Assume that  $(DY / \int_a^b (D_s Y)^2 ds) I_{[a,b]}$  belongs to  $\text{Dom } \delta$ . Then  $Y$  has an absolutely continuous distribution with density  $p$  that satisfies the inequality*

$$p(x) \leq E \left| \int_a^b \left( \frac{D_s Y}{\int_a^b (D_s Y)^2 ds} \right) dW_s \right|.$$

Proof: It follows from Proposition 1 and (2.6) in Moret and Nualart (2000).  $\square$

The following Proposition is also a slight modification of Corollary 2 of Moret-Nualart (2000).

**Proposition 1.2** *Let  $Y$  be a random variable in the space  $\mathbb{D}^{1,2}$  such that  $\int_0^1 (D_s Y)^2 ds > 0$  a.s. Let  $Z$  be a positive square integrable random variable such that  $(ZDY / \int_0^1 (D_s Y)^2 ds) I_{[0,1]}$  belongs to  $\text{Dom } \delta$ . Then, for any  $f \in L^2(\mathbb{R})$ , we have*

$$|E(f(Y)^2 Z)| = \|f\|_2^2 E \left| \delta \left( \frac{ZDY}{\|DY\|_H^2} \right) \right|.$$

Proof: See Corollary 2 in Moret-Nualart (2000). The same proof works using a dominated convergence argument.  $\square$

**Lemma 1.3** *Fix  $p \geq 1$ . Suppose that  $u$  satisfies hypotheses **(H1)** and **(H2)**. Let  $Z \in \mathbb{D}^{1,2p}$ . Then, we have, for  $0 \leq a < b \leq 1$ :*

$$\begin{aligned} & E \left| \int_a^b Z \frac{D_t X_b}{\int_a^b (D_t X_b)^2 dt} dW_t \right|^p \\ & \leq C_0 (b-a)^{-p/2} \left( (E|Z|^{2p})^{1/2} + \left( E \left| \int_a^b (D_t Z)^2 dt \right|^p \right)^{1/2} \right), \end{aligned}$$

where  $C_0$  is a constant does not depend on  $Z$ .

Proof: See Lemma 10 in Moret-Nualart (2000).  $\square$

## 2 Stochastic integration with respect to local time of the martingale

Following the ideas of Eisenbaum (2000), we consider first the space of functions for whose elements we can define a stochastic integration with respect to local time.

Let  $f$  be a measurable function from  $\mathbb{R} \times [0, 1]$  into  $\mathbb{R}$ . We define the norm  $\|\cdot\|$  by

$$\|f\| = \left( \int_0^1 \int_{\mathbb{R}} f^2(x, s) \frac{1}{s^{3/4}} dx ds \right)^{1/2}$$

Consider the set of functions

$$\mathcal{H} = \{f : \|f\| < +\infty\}.$$

It is easy to check that  $\mathcal{H}$  is a Banach space.

Let us consider  $X$  a nondegenerate martingale of the type  $X_t = \int_0^t u_s dW_s$  where  $u$  is an adapted stochastic process satisfying hypotheses **(H1)** and **(H2)**. Let us show now how to define a stochastic integration over the plane with respect to the local time  $L$  of the process  $X$  for the elements of  $\mathcal{H}$ .

Let  $f_\Delta$  be an elementary function,

$$f_\Delta(x, s) := \sum_{(x_k, s_l) \in \Delta} f_{kl} I_{(x_k, x_{k+1}]}(x) I_{(s_l, s_{l+1}]}(s),$$

where  $(x_k)_{1 \leq k \leq m_1}$  is a finite sequence of real numbers,  $(s_l)_{1 \leq l \leq m_2}$  is a subdivision of  $[0, 1]$ ,  $(f_{kl})_{1 \leq k \leq m_1; 1 \leq l \leq m_2}$  is a sequence of real numbers and finally,  $\Delta = \{(x_k, s_l), 1 \leq k \leq m_1, 1 \leq l \leq m_2\}$ . It is easy to check that the elementary functions are dense in  $\mathcal{H}$ .

We define the integration for the elementary function  $f_\Delta$  with respect to the local time  $L$  of the martingale  $X$  as follows

$$\int_0^1 \int_{\mathbb{R}} f_\Delta(x, s) dL_s^x = \sum_{(x_k, s_l) \in \Delta} f_{kl} (L_{s_{l+1}}^{x_{k+1}} - L_{s_l}^{x_{k+1}} - L_{s_{l+1}}^{x_k} + L_{s_l}^{x_k}).$$

Let  $f$  be a function of  $\mathcal{H}$ . Let us consider  $(f_n)_{n \in \mathbb{N}}$  a sequence of elementary functions converging to  $f$  in  $\mathcal{H}$ . We will check that the sequence  $\left( \int_0^1 \int_{\mathbb{R}} f_n(x, s) dL_s^x \right)_{n \in \mathbb{N}}$  converges in  $L^1$  and that the limit does not depend on the choice of the sequence  $(f_n)_{n \in \mathbb{N}}$ . So, we will use this limit as the definition of the integral  $\int_0^1 \int_{\mathbb{R}} f(x, s) dL_s^x$ .

First of all, let us see a previous lemmas.

**Lemma 2.1** *For any locally bounded Borel measurable function  $f$  and any  $t \in (0, 1]$  we have*

$$\int_{\mathbb{R}} f(a) d_a L_t^a = -[f(X), X]_t,$$

where  $d_a L_t^a$  denotes the integral with respect to  $a \longrightarrow L_t^a$ .

Proof: It follows easily from Theorem 0.3 and Theorem 0.1. □

**Lemma 2.2** *Consider  $f_1(x) := I_{(a, b]}(x)$  and  $f_2(x) := I_{(c, d]}(x)$ , where  $a < b$  and  $c < d$  are real numbers. Then for all  $t_i < t_j \leq t$ ,*

$$\begin{aligned} & E [f_1(X_{t_{i+1}}) f_2(X_{t_{j+1}}) (X_{t_{i+1}} - X_{t_i}) (X_{t_{j+1}} - X_{t_j})] \\ & \leq E [f_1(X_{t_{i+1}}) f_2(X_{t_{j+1}}) C_{ij}], \end{aligned}$$

where

$$\|C_{ij}\|_2 \leq C \frac{(t_{i+1} - t_i)(t_{j+1} - t_j)}{\sqrt{t_{i+1}(t_{j+1} - t_{i+1})}},$$

and  $C$  does not depend on  $f_1$  and  $f_2$ .

Proof: When  $f_1 = f_2 = f \in \mathcal{C}_K^\infty(\mathbb{R})$ , this inequality is checked in the proof of Proposition 14 in Moret and Nualart (2000). The same proof also works when  $f_1 \neq f_2$  with  $f_1, f_2 \in \mathcal{C}_K^\infty(\mathbb{R})$ . Now, fixed our functions  $f_1, f_2$  let us consider sequences  $f_1^n \uparrow f_1$  and  $f_2^n \uparrow f_2$  with  $f_n^i \in \mathcal{C}_K^\infty(\mathbb{R})$  for all  $n$  and  $i \in \{1, 2\}$ . Then, the result can be obtained by a dominated convergence argument.  $\square$

**Theorem 2.3** *Let  $f$  be a function of  $\mathcal{H}$ . Then, there exists the integral  $\int_0^t \int_{\mathbb{R}} f(x, s) dL_s^x$  for any  $t \in [0, 1]$ .*

Proof: Let  $f_\Delta$  be an elementary function. From Theorem 0.2 and Lemma 2.1 it is easy to get that the quadratic covariation  $[f(X, \cdot), X]_t$  exists and that

$$\int_0^t \int_{\mathbb{R}} f_\Delta(x, s) dL_s^x = -[f_\Delta(X, \cdot), X]_t.$$

The key of the proof is to check that for all elementary function  $f_\Delta$

$$E \left( \left| \int_0^t \int_{\mathbb{R}} f_\Delta(x, s) dL_s^x \right| \right) \leq C \|f_\Delta\|, \quad (2)$$

where the constant does not depend on  $f_\Delta$ .

Notice that,

$$\begin{aligned} & E \left( \left| \int_0^t \int_{\mathbb{R}} f_\Delta(x, s) dL_s^x \right| \right) \\ &= E(|[f_\Delta(X, \cdot), X]_t|) \\ &= E \left( \left| \lim_{n \rightarrow \infty} \sum_{t_i \in D_n, t_i \leq t} (f_\Delta(X_{t_{i+1}}, t_{i+1}) - f_\Delta(X_{t_i}, t_i)) (X_{t_{i+1}} - X_{t_i}) \right| \right) \\ &\leq \left\{ 2 \liminf_{n \rightarrow \infty} E \left( \left| \sum_{t_i \in D_n, t_i \leq t} f_\Delta(X_{t_{i+1}}, t_{i+1}) (X_{t_{i+1}} - X_{t_i}) \right|^2 \right) \right. \\ &\quad \left. + 2 \liminf_{n \rightarrow \infty} E \left( \left| \sum_{t_i \in D_n, t_i \leq t} f_\Delta(X_{t_i}, t_i) (X_{t_{i+1}} - X_{t_i}) \right|^2 \right) \right\}^{\frac{1}{2}} \\ &:= \left( 2 \left( \liminf_{n \rightarrow \infty} I_1 \right) + 2 \left( \liminf_{n \rightarrow \infty} I_2 \right) \right)^{\frac{1}{2}}, \end{aligned} \quad (3)$$

where in the last inequality we have used Fatou's lemma.

Along the study of  $I_1$  and  $I_2$  we will make use of the methods presented in the proofs of Propositions 13 and 14 in Moret and Nualart (2000). For the sake of completeness, we will give the main steps of our proofs in the study of  $I_2$ . For the other terms, we will refer the reader to the paper of Moret and Nualart (2000).

By the isometry, and using Propositions 1.2 and 1.3

$$\begin{aligned}
I_2 &= E \left( \sum_{t_i \in D_n, t_i \leq t} f_{\Delta}^2(X_{t_i}, t_i) \int_{t_i}^{t_{i+1}} u_s^2 ds \right) \\
&= \sum_{t_i \in D_n, t_i \leq t} E \left( f_{\Delta}^2(X_{t_i}, t_i) \int_{t_i}^{t_{i+1}} u_s^2 ds \right) \\
&\leq \sum_{t_i \in D_n, t_i \leq t} \int_{\mathbb{R}} f_{\Delta}^2(x, t_i) dx E \left| \delta \left( \frac{(\int_{t_i}^{t_{i+1}} u_s^2 ds) DX_{t_i}}{\|DX_{t_i}\|_H^2} \right) \right| \\
&\leq C \sum_{t_i \in D_n, t_i \leq t} \int_{\mathbb{R}} f_{\Delta}^2(x, t_i) dx t_i^{-\frac{1}{2}} \left( \sqrt{E \left| \int_{t_i}^{t_{i+1}} u_s^2 ds \right|^2} + \sqrt{E \int_0^{t_i} \left( D_t \left( \int_{t_i}^{t_{i+1}} u_s^2 ds \right) \right)^2 dt} \right) \\
&= C \sum_{t_i \in D_n, t_i \leq t} \int_{\mathbb{R}} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} f_{kl}^2 I_{(x_k, x_{k+1}]}(x) I_{(s_l, s_{l+1}]}(t_i) t_i^{-\frac{1}{2}} (t_{i+1} - t_i) dx \\
&= C \int_{\mathbb{R}} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} f_{kl}^2 I_{(x_k, x_{k+1}]}(x) \left( \sum_{t_i \in D_n, t_i \leq t} I_{(s_l, s_{l+1}]}(t_i) t_i^{-\frac{1}{2}} (t_{i+1} - t_i) \right) dx \\
&= C \int_{\mathbb{R}} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} f_{kl}^2 I_{(x_k, x_{k+1}]}(x) \left( \int_0^t \sum_{t_i \in D_n, t_i \leq t} I_{(s_l, s_{l+1}]}(t_i) t_i^{-\frac{1}{2}} I_{(t_i, t_{i+1}]}(s) ds \right) dx.
\end{aligned}$$

Using the condition **(M)** over the partitions, we have that, by bounded convergence,

$$\lim_{n \rightarrow \infty} \int_0^t \sum_{t_i \in D_n, t_i \leq t} I_{(s_l, s_{l+1}]}(t_i) t_i^{-\frac{1}{2}} I_{(t_i, t_{i+1}]}(s) ds = \int_0^t I_{(s_l, s_{l+1}]}(s) s^{-\frac{1}{2}} ds,$$

and then,

$$\begin{aligned}
\liminf_{n \rightarrow \infty} I_2 &\leq C \int_{\mathbb{R}} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} f_{kl}^2 I_{(x_k, x_{k+1}]}(x) \int_0^t I_{(s_l, s_{l+1}]}(s) s^{-\frac{1}{2}} ds dx \\
&= C \int_0^t \int_{\mathbb{R}} f_{\Delta}^2(x, s) \frac{1}{\sqrt{s}} dx ds \\
&\leq C \|f_{\Delta}\|^2.
\end{aligned} \tag{4}$$



On the other hand,

$$\begin{aligned}
I_1 &= E \left( \left| \sum_{t_i \in D_n, t_i \leq t} f_\Delta(X_{t_{i+1}}, t_{i+1}) (X_{t_{i+1}} - X_{t_i}) \right|^2 \right) \\
&= E \left( \sum_{t_i \in D_n, t_i \leq t} f_\Delta^2(X_{t_{i+1}}, t_{i+1}) (X_{t_{i+1}} - X_{t_i})^2 \right) \\
&\quad + 2E \left( \sum_{t_i, t_j \in D_n, t_i < t_j \leq t} f_\Delta(X_{t_{i+1}}, t_{i+1}) f_\Delta(X_{t_{j+1}}, t_{j+1}) (X_{t_{i+1}} - X_{t_i}) (X_{t_{j+1}} - X_{t_j}) \right) \\
&:= I_{1,1} + 2I_{1,2}.
\end{aligned} \tag{5}$$

Following now the methods of Proposition 14 of Moret and Nualart (1999) and using again Propositions 1.2 and 1.3 as we did in the study of  $I_2$ , we get that

$$I_{1,1} \leq C \sum_{t_i \in D_n, t_i \leq t} \int_{\mathbb{R}} f_\Delta^2(x, t_i) dx t_{i+1}^{-\frac{1}{2}} (t_{i+1} - t_i).$$

By similar computations to those of the term  $I_2$  we obtain that

$$\liminf_{n \rightarrow \infty} I_{1,1} \leq C \|f_\Delta\|^2. \tag{6}$$

Let us study now  $I_{1,2}$ . Using Lemma 2.2, notice that

$$\begin{aligned}
I_{1,2} &= E \left( \sum_{t_i, t_j \in D_n, t_i < t_j \leq t} f_\Delta(X_{t_{i+1}}, t_{i+1}) f_\Delta(X_{t_{j+1}}, t_{j+1}) (X_{t_{i+1}} - X_{t_i}) (X_{t_{j+1}} - X_{t_j}) \right) \\
&\leq \sum_{t_i, t_j \in D_n, t_i < t_j \leq t} E (f_\Delta(X_{t_{i+1}}, t_{i+1}) f_\Delta(X_{t_{j+1}}, t_{j+1}) C_{ij}).
\end{aligned}$$

Following again the methods of the proof of Proposition 14 of Moret and Nualart (1999) -more precisely, the proof of inequalities (5.36) and (5.37)- the last expression is bounded by

$$\begin{aligned}
&\sum_{t_i, t_j \in D_n, t_i < t_j \leq t} E (f_\Delta^2(X_{t_{i+1}}, t_{i+1}) f_\Delta^2(X_{t_{j+1}}, t_{j+1}))^{\frac{1}{2}} E (C_{ij}^2)^{\frac{1}{2}} \\
&\leq C \sum_{t_i, t_j \in D_n, t_i < t_j \leq t} \frac{(t_{i+1} - t_i)(t_{j+1} - t_j)}{(t_{i+1}(t_{j+1} - t_{i+1}))^{\frac{3}{4}}} \left( \int_{\mathbb{R}} f_\Delta^2(x, t_{i+1}) dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} f_\Delta^2(x, t_{j+1}) dx \right)^{\frac{1}{2}} \\
&\leq C \sum_{t_i, t_j \in D_n, t_i < t_j \leq t} \frac{(t_{i+1} - t_i)(t_{j+1} - t_j)}{(t_{i+1}(t_{j+1} - t_{i+1}))^{\frac{3}{4}}} \left( \int_{\mathbb{R}} f_\Delta^2(x, t_{i+1}) dx + \int_{\mathbb{R}} f_\Delta^2(x, t_{j+1}) dx \right) \\
&:= C(I_{1,2,1} + I_{1,2,2}).
\end{aligned} \tag{7}$$

Since

$$\sum_{t_j \in D_n, t_i < t_j \leq t} \frac{(t_{j+1} - t_j)}{(t_{j+1} - t_{i+1})^{\frac{3}{4}}} = \int_{t_{i+1}}^t \sum_{t_j \in D_n, t_i < t_j \leq t} \frac{1}{(t_{j+1} - t_{i+1})^{\frac{3}{4}}} I_{(t_j, t_{j+1}]}(s) ds,$$

we get that

$$\begin{aligned}
I_{1,2,1} &\leq \sum_{t_i \in D_n, t_i < t_j \leq t} \frac{(t_{i+1} - t_i)}{t_{i+1}^{\frac{3}{4}}} \int_{t_{i+1}}^t \frac{1}{(s - t_{i+1})^{\frac{3}{4}}} ds \left( \int_{\mathbb{R}} f_{\Delta}^2(x, t_{i+1}) dx \right) \\
&\leq C \sum_{t_i \in D_n, t_i < t_j \leq t} \frac{(t_{i+1} - t_i)}{t_{i+1}^{\frac{3}{4}}} \int_{\mathbb{R}} f_{\Delta}^2(x, t_{i+1}) dx.
\end{aligned}$$

And this clearly yields that

$$\liminf_{n \rightarrow \infty} I_{1,2,1} \leq C \int_0^t \int_{\mathbb{R}} f_{\Delta}^2(x, s) \frac{1}{s^{\frac{3}{4}}} ds = C \|f_{\Delta}\|^2. \quad (8)$$

Finally we have to consider  $I_{1,2,2}$ . First of all, notice that

$$\begin{aligned}
I_{1,2,2} &:= \sum_{t_i, t_j \in D_n, t_i < t_j \leq t} \frac{(t_{i+1} - t_i)(t_{j+1} - t_j)}{(t_{i+1}(t_{j+1} - t_{i+1}))^{\frac{3}{4}}} \left( \int_{\mathbb{R}} f_{\Delta}^2(x, t_{j+1}) dx \right) \\
&= \sum_{t_i, t_j \in D_n, t_i < t_j \leq t} \frac{(t_{i+1} - t_i)(t_{j+1} - t_j)}{(t_{i+1}(t_{j+1} - t_{i+1}))^{\frac{3}{4}}} \left( \int_{\mathbb{R}} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} f_{kl}^2 I_{(x_k, x_{k+1}]}(x) I_{(s_l, s_{l+1}]}(t_{j+1}) dx \right) \\
&\leq \int_{\mathbb{R}} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} f_{kl}^2 I_{(x_k, x_{k+1}]}(x) \sum_{t_i \in D_n, t_i < t} \frac{(t_{i+1} - t_i)}{t_{i+1}^{\frac{3}{4}}} \\
&\quad \times \sum_{t_j \in D_n, t_i < t_j \leq t} \int_{t_{i+1}}^t \frac{1}{(s - t_{i+1})^{\frac{3}{4}}} I_{(s_l, s_{l+1}]}(t_{j+1}) I_{(t_j, t_{j+1}]}(s) ds dx
\end{aligned}$$

From the obvious inequality

$$I_{(s_l, s_{l+1}]}(t_{j+1}) I_{(t_j, t_{j+1}]}(s) \leq I_{(s_l, s_{l+1}]}(s) I_{(t_j, t_{j+1}]}(s) + I_{(t_j, s_l]}(s) I_{(t_j, t_{j+1}]}(s_l),$$

we obtain the bound

$$I_{1,2,2} \leq I_{1,2,2,1} + I_{1,2,2,2}, \quad (9)$$

where

$$\begin{aligned}
I_{1,2,2,1} &= \int_{\mathbb{R}} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} f_{kl}^2 I_{(x_k, x_{k+1}]}(x) \\
&\quad \times \sum_{t_i \in D_n, t_i < t} \frac{(t_{i+1} - t_i)}{t_{i+1}^{\frac{3}{4}}} \sum_{t_j \in D_n, t_i < t_j \leq t} \int_{t_{i+1}}^t \frac{1}{(s - t_{i+1})^{\frac{3}{4}}} I_{(s_l, s_{l+1}]}(s) I_{(t_j, t_{j+1}]}(s) ds dx \\
I_{1,2,2,2} &= \int_{\mathbb{R}} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} f_{kl}^2 I_{(x_k, x_{k+1}]}(x) \\
&\quad \times \sum_{t_i \in D_n, t_i < t} \frac{(t_{i+1} - t_i)}{t_{i+1}^{\frac{3}{4}}} \sum_{t_j \in D_n, t_i < t_j \leq t} \int_{t_{i+1}}^t \frac{1}{(s - t_{i+1})^{\frac{3}{4}}} I_{(t_j, s_l]}(s) I_{(t_j, t_{j+1}]}(s_l) ds dx.
\end{aligned}$$

Now, since we can write

$$I_{1,2,2,1} = \int_{\mathbb{R}} \sum_{t_i \in D_n, t_i < t} \frac{(t_{i+1} - t_i)}{t_{i+1}^{\frac{3}{4}}} \int_{t_{i+1}}^t f_{\Delta}^2(x, s) \frac{1}{(s - t_{i+1})^{\frac{3}{4}}} ds dx,$$

using an argument of bounded convergence we have that

$$\begin{aligned} \liminf_{n \rightarrow \infty} I_{1,2,2,1} &\leq \int_{\mathbb{R}} \int_0^t \frac{1}{u^{\frac{3}{4}}} \int_u^t f_{\Delta}^2(x, s) \frac{1}{(s - u)^{\frac{3}{4}}} ds du dx \\ &= \int_{\mathbb{R}} \int_0^t f_{\Delta}^2(x, s) \int_0^s \frac{1}{u^{\frac{3}{4}}} \frac{1}{(s - u)^{\frac{3}{4}}} du ds dx \\ &\leq C \int_{\mathbb{R}} \int_0^t f_{\Delta}^2(x, s) \frac{1}{s^{\frac{1}{2}}} ds dx \\ &\leq C \|f_{\Delta}\|^2. \end{aligned} \tag{10}$$

On the other hand, observe that fixed  $l$ , there exists only one  $j$  (that we will denote by  $j(l)$ ) such that  $t_{j(l)} < s_l \leq t_{j(l)+1}$ . So,

$$\begin{aligned} &\sum_{t_i, t_j \in D_n, t_i < t_j \leq t} \frac{(t_{i+1} - t_i)}{t_{i+1}^{\frac{3}{4}}} \int_{t_{i+1}}^t \frac{1}{(s - t_{i+1})^{\frac{3}{4}}} I_{(t_j, s_l]}(s) I_{(t_j, t_{j+1}]}(s_l) ds \\ &\leq \sum_{t_i \in D_n, t_i < t_{j(l)} \leq t} \frac{(t_{i+1} - t_i)}{t_{i+1}^{\frac{3}{4}}} \int_{t_{j(l)}}^{t_{j(l)+1}} \frac{1}{(s - t_{i+1})^{\frac{3}{4}}} ds. \end{aligned}$$

Now, using that for  $i < j(l)$

$$\int_{t_{j(l)}}^{t_{j(l)+1}} \frac{1}{(s - t_{i+1})^{\frac{3}{4}}} ds \leq \int_{t_{j(l)}}^{t_{j(l)+1}} \frac{1}{(s - t_{j(l)})^{\frac{3}{4}}} ds \leq 4|D_n|^{\frac{1}{4}},$$

and that

$$\sum_{t_i \in D_n, t_i < t_{j(l)} \leq t} \frac{(t_{i+1} - t_i)}{t_{i+1}^{\frac{3}{4}}} \leq \int_0^1 \frac{1}{s^{\frac{3}{4}}} ds < \infty,$$

we obtain easily that

$$\lim_{n \rightarrow \infty} I_{1,2,2,2} = 0. \tag{11}$$

So, putting together (3)-(11), we have proved (2).

Now, given  $f \in \mathcal{H}$ , let us consider  $\{f_n\}_{n \in \mathbb{N}}$  a sequence of elementary functions converging to  $f$  in  $\mathcal{H}$ , and we define

$$\int_0^t \int_{\mathbb{R}} f(x, s) dL_s^x = L^1 - \lim_{n \rightarrow \infty} \left( \int_0^t \int_{\mathbb{R}} f_n(x, s) dL_s^x \right).$$

Clearly, this limit exists. Indeed, for any  $\varepsilon > 0$  there exists  $n_0$  such that for any  $n, m \geq n_0$ ,  $\|f_n - f_m\| < \varepsilon$  and using inequality (2) we obtain that

$$E \left| \int_0^t \int_{\mathbb{R}} f_n(x, s) dL_s^x - \int_0^t \int_{\mathbb{R}} f_m(x, s) dL_s^x \right| = E \left| \int_0^t \int_{\mathbb{R}} (f_n(x, s) - f_m(x, s)) dL_s^x \right| \leq \|f_n - f_m\| < \varepsilon.$$

Moreover, using again inequality (2), it is clear that the definition does not depend on the choice of the sequence  $(f_n)$ . Indeed, given  $(f_n^1)_{n \in \mathbb{N}}$  and  $(f_n^2)_{n \in \mathbb{N}}$  two sequences converging to  $f$  in  $\mathcal{H}$ , we have

$$E \left( \left| \int_0^t \int_{\mathbb{R}} f_n^1(x, s) dL_s^x - \int_0^t \int_{\mathbb{R}} f_n^2(x, s) dL_s^x \right| \right) \leq \|f_n^1 - f_n^2\| \leq \|f_n^1 - f\| + \|f - f_n^2\|,$$

that goes to zero when  $n$  tends to infinity. □

**Remark 2.4** *If  $f$  satisfies condition (C), from Theorem 0.2 we know that the quadratic covariation  $[f(X, \cdot), X]$  exists. Moreover, if  $f \in \mathcal{H}$ , from the uniqueness of the extension in the construction of the integral in Theorem 2.3 we get that*

$$\int_0^t \int_{\mathbb{R}} f(x, s) dL_s^x = -[f(X, \cdot), X]_t.$$

The following results is an obvious consequence of Theorem 0.2 and Remark 2.4.

**Corollary 2.5** *Let  $u$  be a process satisfying (H1) and (H2). Set  $X = \int_0^t u s dW_s$ . Consider a sequence of partitions  $D_n$  of partitions of  $[0, 1]$  verifying conditions (M). Let  $F(x, t)$  be an absolutely continuous function in  $x$  such that the partial derivative  $f(\cdot, t)$  satisfies (C). Then, if  $f \in \mathcal{H}$ , we have the following extension for the Itô's formula:*

$$F(X_t, t) = F(0, 0) + \int_0^t f(X_s, s) dX_s - \frac{1}{2} \int_0^t \int_{\mathbb{R}} f(x, s) dL_s^x + \int_0^t F(X_s, ds).$$

### 3 Itô's formula extension

Now we can state the main result of this paper.

#### Theorem 3.1

*Hypotheses over the martingale:*

1. *Let  $u$  be an adapted process satisfying (H1) and (H2). Set  $X = \int_0^t u_s dW_s$ .*

*Hypotheses over the function:*

1. *Let  $F$  be a function defined on  $\mathbb{R} \times [0, 1]$  such that  $F$  admits first order Radon-Nikodym derivatives with respect to each parameter.*
2. *Assume that these derivatives satisfy that for every  $A \in \mathbb{R}$ ,*

$$\begin{aligned} \int_0^1 \int_{-A}^A \left| \frac{\partial F}{\partial t}(x, s) \right| dx \frac{1}{\sqrt{s}} ds &< +\infty \\ \int_0^1 \int_{-A}^A \left( \frac{\partial F}{\partial x}(x, s) \right)^2 dx \frac{1}{\sqrt{s}} ds &< +\infty. \end{aligned}$$

Then, for all  $t \in [0, 1]$ ,

$$F(X_t, t) = F(0, 0) + \int_0^t \frac{\partial F}{\partial x}(X_s, s) dX_s + \int_0^t \frac{\partial F}{\partial t}(X_s, s) ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) dL_s^x.$$

Proof: Using localization arguments we can assume that  $F$  has compact support and

$$\begin{aligned} \int_0^1 \int_{\mathbb{R}} \left| \frac{\partial F}{\partial t}(x, s) \right| dx \frac{1}{\sqrt{s}} ds &< +\infty \\ \int_0^1 \int_{\mathbb{R}} \left( \frac{\partial F}{\partial x}(x, s) \right)^2 dx \frac{1}{\sqrt{s}} ds &< +\infty. \end{aligned}$$

Let  $g \in \mathcal{C}^\infty$  be a function with compact support from  $\mathbb{R}$  to  $\mathbb{R}^+$  such that  $\int_{\mathbb{R}} g(s) ds = 1$ . We define, for any  $n \in \mathbb{N}$ ,

$$g_n(s) = ng(ns)$$

and

$$F_n(x, t) = \int_0^1 \int_{\mathbb{R}} F(y, s) g_n(t-s) g_n(x-y) dy ds.$$

Then  $F_n \in \mathcal{C}^\infty(\mathbb{R} \times [0, 1])$ . Hence, by the usual Itô's formula, for every  $\varepsilon > 0$ , we can write

$$F_n(X_t, t) = F_n(X_\varepsilon, \varepsilon) + \int_\varepsilon^t \frac{\partial F_n}{\partial x}(X_s, s) dX_s + \int_\varepsilon^t \frac{\partial F_n}{\partial t}(X_s, s) ds + \frac{1}{2} \int_\varepsilon^t u_s^2 \frac{\partial^2 F_n}{\partial x^2}(X_s, s) ds. \quad (12)$$

Using the arguments of Azéma *et al.* (1998) we will study the convergence of (12).

Since  $F$  is a continuous function with compact support, it is easy to check that  $(F_n(X_t, t))_{n \in \mathbb{N}}$  converges in probability to  $F(X_t, t)$ .

On the other hand

$$\int_0^1 \int_{\mathbb{R}} \left| \frac{\partial F}{\partial t}(x, s) \right| dx ds \leq \int_0^1 \int_{\mathbb{R}} \left| \frac{\partial F}{\partial t}(x, s) \right| dx \frac{1}{\sqrt{s}} ds < +\infty.$$

Hence,  $\frac{\partial F}{\partial t} \in L^1(\mathbb{R} \times [0, 1])$ . Under our hypothesis over the martingale  $X$ , it follows from Proposition 1.1 and Lemma 1.3 that for any  $t \in [0, 1]$ , the random variable  $X_t$  is absolutely continuous with density  $p_t$  satisfying the estimate

$$p_t(x) \leq \frac{C}{\sqrt{t}}.$$

Then, it is easy to see that  $\left( \int_\varepsilon^t \frac{\partial F_n}{\partial t}(X_s, s) ds \right)_{n \in \mathbb{N}}$  converges in probability to  $\left( \int_\varepsilon^t \frac{\partial F}{\partial t}(X_s, s) ds \right)$ . Indeed,

$$\begin{aligned} E \left( \left| \int_\varepsilon^t \left( \frac{\partial F_n}{\partial t}(X_s, s) - \frac{\partial F}{\partial t}(X_s, s) \right) ds \right| \right) &\leq \int_\varepsilon^t \int_{\mathbb{R}} \left| \frac{\partial F_n}{\partial t}(x, s) - \frac{\partial F}{\partial t}(x, s) \right| p_s(x) dx ds \\ &\leq C \int_\varepsilon^t \int_{\mathbb{R}} \left| \frac{\partial F_n}{\partial t}(x, s) - \frac{\partial F}{\partial t}(x, s) \right| \frac{1}{\sqrt{s}} dx ds \\ &\leq \frac{C}{\sqrt{\varepsilon}} \int_\varepsilon^t \int_{\mathbb{R}} \left| \frac{\partial F_n}{\partial t}(x, s) - \frac{\partial F}{\partial t}(x, s) \right| dx ds, \end{aligned}$$

that goes to zero, when  $n$  tends to infinity, since  $\frac{\partial F}{\partial t} \in L^1(\mathbb{R} \times [0, 1])$  and

$$\frac{\partial F_n}{\partial t}(x, t) = \int_0^1 \int_{\mathbb{R}} \frac{\partial F}{\partial t}(y, s) g_n(t-s) g_n(x-y) dy ds.$$

Similarly, we can prove that  $\left( \int_{\varepsilon}^t \frac{\partial F_n}{\partial x}(X_s, s) dX_s \right)_{n \in \mathbb{N}}$  converges in probability to  $\left( \int_{\varepsilon}^t \frac{\partial F}{\partial x}(X_s, s) dX_s \right)$ . Indeed, using the same arguments we get that  $\frac{\partial F}{\partial x} \in L^2(\mathbb{R} \times [0, 1])$ . Then,

$$\begin{aligned} & E \left( \left| \int_{\varepsilon}^t \left( \frac{\partial F_n}{\partial x}(X_s, s) - \frac{\partial F}{\partial x}(X_s, s) \right) dX_s \right|^2 \right) \\ &= E \left( \left| \int_{\varepsilon}^t \left( \frac{\partial F_n}{\partial x}(X_s, s) - \frac{\partial F}{\partial x}(X_s, s) \right) u_s dW_s \right|^2 \right) \\ &= E \left( \int_{\varepsilon}^t \left( \frac{\partial F_n}{\partial x}(X_s, s) - \frac{\partial F}{\partial x}(X_s, s) \right)^2 u_s^2 ds \right). \end{aligned}$$

Following the same ideas of Proposition 12 in Moret and Nualart (2000), Proposition 1.2 and Lemma 1.3 yield the following bound for the last expression

$$\begin{aligned} & C \int_{\varepsilon}^t \int_{\mathbb{R}} \left( \frac{\partial F_n}{\partial x}(x, s) - \frac{\partial F}{\partial x}(x, s) \right)^2 \frac{1}{\sqrt{s}} dx ds \\ & \leq \frac{C}{\sqrt{\varepsilon}} \int_{\varepsilon}^t \int_{\mathbb{R}} \left( \frac{\partial F_n}{\partial x}(x, s) - \frac{\partial F}{\partial x}(x, s) \right)^2 dx ds \end{aligned} \tag{13}$$

that goes to zero when  $n$  tends to infinity, since  $\frac{\partial F}{\partial x} \in L^2(\mathbb{R} \times [0, 1])$  and

$$\frac{\partial F_n}{\partial x}(x, t) = \int_0^1 \int_{\mathbb{R}} \frac{\partial F}{\partial x}(y, s) g_n(t-s) g_n(x-y) dy ds.$$

So, letting  $n$  to infinity in (12), we get that the sequence

$$\left( \frac{1}{2} \int_{\varepsilon}^t u_s^2 \frac{\partial^2 F_n}{\partial x^2}(X_s, s) ds \right)_{n \in \mathbb{N}}$$

converges in probability to

$$F(X_t, t) - F(X_{\varepsilon}, \varepsilon) - \int_{\varepsilon}^t \frac{\partial F}{\partial x}(X_s, s) dX_s - \int_{\varepsilon}^t \frac{\partial F}{\partial t}(X_s, s) ds.$$

But, since  $\frac{\partial F_n}{\partial x}(x, s) I_{(\varepsilon, t)}(s) \in \mathcal{H}$ , from Theorem 0.2 and Corollary 2.5, we get that

$$\begin{aligned} \int_{\varepsilon}^t u_s^2 \frac{\partial^2 F_n}{\partial x^2}(X_s, s) ds &= \left[ \frac{\partial F_n}{\partial x}(X, \cdot), X \right]_t - \left[ \frac{\partial F_n}{\partial x}(X, \cdot), X \right]_{\varepsilon} \\ &= - \int_0^1 \int_{\mathbb{R}} \frac{\partial F_n}{\partial x}(x, s) I_{(\varepsilon, t)}(s) dL_s^x. \end{aligned}$$

The next step of the proof is to check that  $\left(\frac{\partial F_n}{\partial x}(x, s)I_{(\varepsilon, t)}(s), x \in \mathbb{R}, s \in [0, 1]\right)_{n \in \mathbb{N}}$  converges in  $\mathcal{H}$  to  $\left(\frac{\partial F}{\partial x}(x, s)I_{(\varepsilon, t)}(s), x \in \mathbb{R}, s \in [0, 1]\right)$ . It suffices to notice that,

$$\int_{\varepsilon}^t \int_{\mathbb{R}} \left( \frac{\partial F_n}{\partial x}(x, s) - \frac{\partial F}{\partial x}(x, s) \right)^2 \frac{1}{s^{\frac{3}{4}}} dx ds \leq \frac{1}{\varepsilon^{\frac{3}{4}}} \int_{\varepsilon}^t \int_{\mathbb{R}} \left( \frac{\partial F_n}{\partial x}(x, s) - \frac{\partial F}{\partial x}(x, s) \right)^2 dx ds$$

that converges to zero when  $n$  tends to infinity. Then, we clearly have proved that

$$\left( \int_0^1 \int_{\mathbb{R}} \frac{\partial F_n}{\partial x}(x, s) I_{(\varepsilon, t)}(s) dL_s^x \right)_{n \in \mathbb{N}}$$

converges in  $L^1$  to  $\int_0^1 \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) I_{(\varepsilon, t)}(s) dL_s^x$ .

So, we have that for any  $\varepsilon > 0$

$$F(X_t, t) = F(X_{\varepsilon}, \varepsilon) + \int_{\varepsilon}^t \frac{\partial F}{\partial x}(X_s, s) dX_s + \int_{\varepsilon}^t \frac{\partial F}{\partial t}(X_s, s) ds - \frac{1}{2} \int_0^1 \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) I_{(\varepsilon, t)}(s) dL_s^x. \quad (14)$$

The last steep is to let  $\varepsilon$  to zero. But we need to check that the limit of the stochastic integral exists. Actually, it is enough to show that

$$E \left( \left| \int_0^t \frac{\partial F}{\partial t}(X_s, s) ds \right| \right) < \infty$$

and that

$$E \left( \int_0^t \frac{\partial F}{\partial x}(X_s, s) dX_s \right)^2 < \infty.$$

But,

$$E \left( \left| \int_0^t \frac{\partial F}{\partial t}(X_s, s) ds \right| \right) \leq C \int_0^1 \int_{\mathbb{R}} \left| \frac{\partial F}{\partial t}(x, s) \right| dx \frac{1}{\sqrt{s}} ds < +\infty.$$

On the other hand, following the same type of arguments that in (13), we are able to write

$$\begin{aligned} E \left( \int_0^t \frac{\partial F}{\partial x}(X_s, s) dX_s \right)^2 &= E \left( \int_0^t \left( \frac{\partial F}{\partial x}(X_s, s) \right)^2 u_s^2 ds \right) \\ &\leq C \int_{\mathbb{R}} \int_0^t \left( \frac{\partial F}{\partial x}(x, s) \right)^2 \frac{1}{\sqrt{s}} ds dx < \infty. \end{aligned}$$

Letting  $\varepsilon$  to zero, the proof is finished. □

**Remark 3.2** Notice that under the hypotheses of Theorem 3.1, it is possible that  $\frac{\partial F}{\partial x}$  does not belong to the space  $\mathcal{H}$ . In this case, using the localization arguments, we can always assume that  $\left(\frac{\partial F}{\partial x}(x, s)I_{(\varepsilon, t)}(s), x \in \mathbb{R}, s \in [0, 1]\right)$  belongs to  $\mathcal{H}$  for any  $\varepsilon > 0$  and we can define

$$\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) dL_s^x = \lim_{\varepsilon \rightarrow 0} \int_0^1 \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) I_{(\varepsilon, t)}(s) dL_s^x.$$

This limit exists in probability since all the other limits in (14) exist.

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